Least-squares and pseudo-unitary migration

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THE PROPERTIES

Stolt (1978) defined a constant-velocity migration equation in his first paper on the subject, but did not define a corresponding modeling or "diffraction" equation. The choice has not been entirely obvious. A simple physical argument leads to one choice; numerical concerns lead to another. We shall see that a diffraction that is the adjoint of Stolt migration corresponds to a simple, easily interpreted Green's function. On the other hand, a diffraction that is the least-squares inverse of migration reconstructs the data most accurately.

Rather than choose a single migration, let us choose a linear modeling equation whose physical assumptions appeal to us. Then we can choose desirable properties for the corresponding migration. This paper will not debate the respective merits of various linear modeling programs. Too many questions must be asked first. Is the source isotropic, zero-phase? Do the receivers measure displacement or pressure, vertically or isotropically? What physical parameters are we perturbing—impedances, velocities, moduli, or reflection coefficients? And so on. Once we have a linear equation that creates data from the model, we can choose a mapping from our data to the model, to be called migration.

Programmers tend to choose a migration that is a generalized inverse of diffraction, most often, the least-squares inverse. This choice restores the original data as well as possible from a diffraction. The two operations can be repeated over and over again, without destroying information or degrading the data.

On the other hand, linear perturbation theory encourages a migration that is the adjoint of diffraction. For example a diffraction might find the first-order linear perturbation of a wavefield recorded on the surface due to the perturbation of a buried source (a non-physical source is the so-called "exploding reflector"). An adjoint migration solves the linearized control problem: find the perturbations of buried sources that will explain non-zero perturbations of the data. Calculate the data perturbations corresponding to each buried source, and sum them all together. Such an approach is typical of Born methods. The forward transformation corresponds to a matrix of Frechet derivatives that linearize the physical modeling of the data. Reversing the direction of perturbations merely transposes the matrix.

I shall begin with the adjoint migration and then derive a least-squares alternative. $G(x,\tau)$ be the earth model, a function of horizontal and vertical spatial coordinates, x and $\tau = depth/velocity$. Let $D(x,t)$ be the corresponding data, a function of x and time t.

$$D = WG \; ; \quad MD = W^* D \quad . \tag{1}$$

where W is diffraction, and M is an adjoint migration. The asterisk indicates the adjoint (or transpose of real matrices). If M corresponds to Stolt's original definition of migration, then W will correspond to the familiar spherical, isotropic Green's function. However, the adjoint migration M is a suboptimum inverse, as we shall see.

When implemented in the frequency domain, the adjoint Stolt transformations destroy certain information: 1) D may contain noise at high dips, requiring imaginary frequencies in G, frequencies which we do not save; 2) Stolt diffraction does not sample the depth-frequency axis k_τ sufficiently when discrete transforms are used, with the result that late diffraction hyperbolas wrap around to the top—migration treats these wraparounds as shallow events. Both D and G may contain information not expressed in the other domain. The second limitation may be ignored by padding zeros onto geologic models. We shall examine integral transforms, so only the first limitation will appear explicitly. Thus diffraction may be exactly inverted, but not migration.

Again, let us begin by assuming knowledge of the forward modeling operator, W (diffraction). In particular let this diffraction be the same as used in equation (1) (the adjoint of Stolt's original definition of migration). Let the least-squares migration, M_{ls}, be the left inverse of W, where

$$M_{ls} W = I, \tag{2}$$

$$\text{but } W M_{ls} = A \neq I. \tag{3}$$

I is the identity operator; A is an operator with zeros and ones on the diagonal in the frequency domain. Thus migration can invert diffraction but not vice versa. The solution of $D = WG$ for G is an over-determined problem. The least-squares solution for

G given W and D defines migration as

$$G = [(W^* W)^{-1} W^*] D \equiv M_{ls} D \ , \qquad (4)$$

We shall show that our chosen diffraction and its least-squares migration satisfy

$$M_{ls} = B_1 W^* , \text{ where } B_1 \text{ is diagonal over } k_x \text{ and } k_\tau \text{ and is invertible,} \qquad (5)$$

$$\text{and } W = B_2 M_{ls}^* , \text{ where } B_2 \text{ is diagonal over } k_x \text{ and } \omega \text{ and is not invertible.} \qquad (6)$$

In the frequency domain the diagonal elements of B_1 and B_2 are real and contain the so-called cosine corrections. Except for these corrections, least-squares migration is the adjoint of forward modeling. Equations (2) and (5) being satisfied by Stolt's migration and diffraction directly shows (4) to be satisfied—migration is a least-squares solution of (1).

DEMONSTRATION OF THE PROPERTIES

Let $\overline{G}$ and $\overline{D}$ be the Fourier transforms of G and D. Let $\overline{W}$ and $\overline{M}$ be the usual Stolt transforms in this new domain.

$$\overline{M} = FMF^* \ ; \qquad \overline{W} = FWF^* \qquad (7)$$

F is the forward Fourier transform. Let the adjoint migration be equivalent to Stolt's definition migration.

$$\overline{M}\ \overline{D} = \overline{W}^* \overline{D} \qquad (8)$$

$$= \frac{| k_\tau |}{\sqrt{v^2 k_x^2 + k_\tau^2}} \int_{-\infty}^{\infty} \overline{D}(\omega, k_x) \delta(\omega - sign(k_\tau) \sqrt{v^2 k_x^2 + k_\tau^2}\ ) \, d\omega \ .$$

v is the constant velocity. The cosine factor before the integral may be regarded as a diagonal operator over k_x and k_τ. Recall that

$$\delta(f(x)) = \sum_n \frac{1}{| f'(x_n) |} \delta(x - x_n) \ ,$$

where x_n are the zeros of $f(x)$. The adjoint of M corresponds to a common definition of diffraction:

$$\overline{D}(\omega,k_x) = \overline{W}\ \overline{G} = \overline{M}^*\ \overline{G} \tag{9}$$

$$= H(\omega^2 - v^2k_x^2) \int_{-\infty}^{\infty} \overline{G}(k_\tau,\ k_x)\delta(k_\tau - sign(\omega)\sqrt{\omega^2 - v^2k_x^2})\ dk_\tau\ ,$$

$$where\ \ H(x) \equiv \begin{cases} 1 & x \geq 1 \\ 0 & x < 1 \end{cases} .$$

The derivative of the argument of the delta function and the cosine factor have neatly canceled. The Heaviside function $H(x)$ is necessary because (8) transforms only to real k_τ.

Stolt did not define a diffraction algorithm in his original paper, but equation (9) is a very appealing choice. This modeling equation corresponds to isotropic pressure sources at depth at isotropic receivers at the surface. Equation (9) implements the familiar spherical Green's function. Let us choose this diffraction as the fixed transformation and find an alternative migration as its least-square's inverse. We propose the following:

$$\overline{G}(k_\tau,k_x) = \overline{M}_{ls}\,\overline{D} = \int_{-\infty}^{\infty} \overline{D}(\omega,\ k_x)\delta(\omega - sign(k_\tau)\sqrt{v^2k_x^2 + k_\tau^2}\ )\ d\omega\ . \tag{10}$$

Notice that this definition contains no obliquity or cosine factor. The adjoint of the least-squares migration is

$$\overline{M}_{ls}^*\overline{G} = \frac{|\,\omega\,|}{\sqrt{\omega^2 - v^2k_x^2}} H(\omega^2 - v^2k_x^2) \int_{-\infty}^{\infty} \overline{G}(k_\tau,\ k_x)\delta(k_\tau - sign(\omega)\sqrt{\omega^2 - v^2k_x^2})\ dk_\tau\ . \tag{11}$$

We observe that

$$\overline{M}_{ls}\,\overline{W}\ \ \overline{G}(k_\tau,\ k_x) = \overline{G}(k_\tau,\ k_x)\ , \tag{12}$$

but

$$\overline{W}\ \ \overline{M}_{ls}\,\overline{D}(\omega,\ k_x) = H(\omega^2 - v^2k_x^2)\overline{D}(\omega,\ k_x). \tag{13}$$

We have verified (2) and (3): diffraction is invertible, migration is not. Let

$$\overline{B}_1 = FB_1F^* \qquad \overline{B}_2 = FB_2F^*\ . \tag{14}$$

Then we observe that

$$\overline{M}_{ls} = \overline{B}_1\overline{W}^* = \overline{B}_1\overline{M} \qquad \overline{W} = \overline{M}^* = \overline{B}_2\overline{M}_{ls}^* \tag{5,6a}$$

where

$$\overline{B}_1 = \frac{\sqrt{v^2k_x^2 + k_\tau^2}}{|\,k_\tau\,|}\ ; \qquad \overline{B}_2 = \frac{\sqrt{\omega^2 - v^2k_x^2}}{|\,\omega\,|} H(\omega^2 - v^2k_x^2)\ . \tag{15}$$

The above transform easily into (5) and (6). Because F is unitary, minimizing the sum of the squares is equivalent in either domain. Now we may prove that our chosen $\overline{M_{ls}}$ is in fact the least-squares inverse of diffraction. By (2) and (5)

$$B_1 W^* W = M_{ls} W = I \quad \rightarrow \quad W^* W = B_1^{-1} \tag{16}$$
$$\rightarrow \quad (W^* W)^{-1} W^* = B_1 W^* = M_{ls} \ .$$

PSEUDO-UNITARY MIGRATION

In our derivation of a least-squares migration, we admitted that the physical $G(x,t)$ had some arbitrariness. It is not a physical object. (It is only a generic measure of "reflectivity" convolved with a spiked wavelet.)

Instead let us construct a W whose adjoint is also the left inverse of W. Redefine the model with the transformation S:

$$\overline{D} = \overline{W}\ \overline{G} = \overline{W} S S^{-1} \overline{G} = \overline{W_u}\ \overline{G_u} \ ; \tag{17}$$
$$\text{where } \ \overline{W_u} \equiv \overline{W}\ S \ \text{ and } \ \overline{G_u} \equiv S^{-1} \overline{G} \ .$$

We want the transpose of W_u to be the left inverse of W_u: $W_u^* W_u = I$. We also require:

$$\overline{G} = (\overline{W}^*\ \overline{W})^{-1} \overline{W}^* \overline{D} \ ; \quad \overline{G_u} = \overline{W_u}^* \overline{D} \tag{18}$$

Equations (17) and (18) imply

$$\overline{G_u} = S^{-1} \overline{G} = S^{-1} (\overline{W}^*\ \overline{W})^{-1} \overline{W}^* \overline{D} = S^*\ \overline{W}^* \overline{D} \tag{19}$$
$$\text{implies} \rightarrow \ S S^* = (\overline{W}^*\ \overline{W})^{-1} = \overline{B}_1$$

The above is satisfied when

$$S = \sqrt{\overline{B}_1} \tag{20}$$

Construct the unitary diffraction as follows:

$$\overline{D}(\omega, k_x) = \overline{W_u}\ \overline{G} \tag{21}$$

$$= \left[\frac{|\omega|}{\sqrt{\omega^2 - v^2 k_x^2}} \right]^{\frac{1}{2}} H(\omega^2 - v^2 k_x^2) \int_{-\infty}^{\infty} \overline{G}(k_\tau, k_x) \delta(k_\tau - sign(\omega) \sqrt{\omega^2 - v^2 k_x^2})\ dk_\tau \ ,$$

The transpose and least-squares inversion define the migration:

$$\overline{G}(k_\tau, k_x) = \overline{M_u}\ \overline{D} = \overline{W_u}^* \overline{D} \tag{22}$$

$$= \left[\frac{|k_\tau|}{\sqrt{v^2 k_x^2 + k_\tau^2}} \right]^{\frac{1}{2}} \int_{-\infty}^{\infty} \overline{D}(\omega, k_x) \delta(\omega - sign(k_\tau) \sqrt{v^2 k_x^2 + k_\tau^2}\ )\ d\omega \ .$$

W_u is unitary because $W_u^* W_u = I$. The migration should be called pseudo-unitary because $W_u W_u^* = M_u^* M_u = A$, where A is diagonal and contains only ones and zeros.

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REFERENCES

Stolt, R.H., 1978, Migration by Fourier transform: Geophysics, **43,** p. 23-48.